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Two-dimensional Yang-Mills theory and moduli spaces of holomorphic differentials

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Abstract

We describe and solve a double scaling limit of large N Yang-Mills theory on a two-dimensional torus. We find the exact strong-coupling expansion in this limit and describe its relation to the conventional Gross-Taylor series. The limit retains only the chiral sector of the full gauge theory and the coefficients of the expansion determine the asymptotic Hurwitz numbers, in the limit of infinite winding number, for simple branched coverings of a torus. These numbers are computed exactly from the gauge theory vacuum amplitude and shown to coincide with the volumes of the principal moduli spaces of holomorphic differentials. The string theory interpretation of the double scaling limit is also described.

1. Introduction. Two-dimensional Yang-Mills theory is an exactly solvable quantum field theory which has over the years found a multitude of both physical and mathematical applications. On the physical side, it is the first example of a gauge theory which can be reformulated precisely as a string theory [1]. It is also deeply connected to integrable systems [2] and conformal field theory [3], and has recently been shown to be equivalent to topological string theory on certain non-compact Calabi-Yau threefolds [4] and to reproduce the counting of instantons in four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry [5]. On the mathematical side, the partition function is the generating function for intersection indices on the symplectic moduli spaces of flat connections on Riemann surfaces [6] and for the orbifold Euler characters of Hurwitz moduli spaces of branched covers with deep links to topological field theory in two dimensions [7].

In this paper we will describe some new applications of Yang-Mills theory on a two-dimensional torus which is based on a double scaling limit of the quantum gauge theory

at large N. Starting from the exact semi-classical expansion of the partition function, we derive the weak-coupling expansion dual to the Gross-Taylor series and compute explicitly the series originally obtained in [3, 8] using the conformal field theory approach. Our approach is based on the exact solutions of certain saddle-point equations in the zero-instanton sector of the two-dimensional gauge theory. We then define a new limit which captures only one chiral sector of the gauge theory and gives a geometrical meaning to the expansions unveiled in [8]. In this limit, only string states of infinite winding number contribute to the vacuum amplitude which thereby becomes the generating function for the asymptotics of Hurwitz numbers of simple branched coverings of a torus. Our saddle-point technique provides a very quick and efficient way to extract exact formulas for these asymptotics which avoids the cumbersome combinatorial techniques usually employed in the mathematics literature. With this realization, we then go on to show that the partition function in the limit actually serves as a generating function for the volumes of moduli spaces of holomorphic differentials on Riemann surfaces. The beauty of this realization is that the observables of two-dimensional Yang-Mills theory, such as Wilson loops and Polyakov lines, could provide generating functions for intersection indices, or other geometrical quantities, on these moduli spaces. We will also briefly describe the string theoretic meaning of the limit and argue that it is more akin to a string representation of noncommutative gauge theory in two dimensions. Further details and applications will be given in [9].

2. Chiral Gross-Taylor series on the torus. The chiral partition function of $U(N)$ Yang-Mills theory on a rectangular torus $\mathbb{T}^2$ of area A is a variant of the usual Migdal strong-coupling expansion given by [1]

$$Z^+(\lambda, N) = \sum_{R\in\mathrm{Rep}^+(U(N))} \mathrm{e}^{-\frac{\lambda}{N}C_2(R)} \ , \tag{1}$$

where $\lambda = g^2 A\, N/2$ is the dimensionless 't Hooft coupling constant and $C_2(R)$ is the quadratic Casimir eigenvalue in the irreducible representation $R = R(Y)$ of the gauge group $U(N)$. The restriction to chiral representations means considering Young tableaux Y with positive number of boxes and dropping the constraint that the number of rows be less than the rank N. In so doing, one essentially assumes that only representations with small numbers of boxes (compared to N) are relevant in the large N limit, the others being exponentially suppressed in N [1]. More generally, a class of representations with box numbers of order N, whose contribution is not exponentially damped, can be found and used to construct the anti-chiral sector. A non-chiral coupled expansion, incorporating both contributions, has been presented in [1] for the case of an $SU(N)$ gauge group and is widely accepted as the complete large N description of the gauge theory.

By using the explicit form of the quadratic Casimir, the sum over Young diagrams may be carried out, and the free energy $F^+(\lambda, N) = \ln Z^+(\lambda, N)$ can be written as the asymptotic $\frac{1}{N}$ expansion [1]

$$F^+(\lambda, N) = \sum_{h=1}^{\infty} \frac{1}{N^{2h-2}}\, F_h^+(\lambda) \ , \quad F_h^+(\lambda) = \lambda^{2h-2} \sum_{n=1}^{\infty} \omega_h^n \ \mathrm{e}^{-n\,\lambda} \ . \tag{2}$$

The non-negative integers ω_h^n are called simple Hurwitz numbers and they give the number of (topological classes of) n-sheeted holomorphic covering maps without folds to the torus

$\mathbb{T}^2$. They count the number of maps from a closed, connected and oriented Riemann surface of genus h to the torus with winding number n and $2h-2$ simple branch points. This expansion contains a Nambu-Goto factor $\mathrm{e}^{-n\lambda}$ as well as a volume factor λ^{2h-2} from the moduli space integration over the positions of the branch points. In other words, the strong coupling expansion for the chiral free energy of $U(N)$ gauge theory on $\mathbb{T}^2$ in the 't Hooft limit is the generating function for the simple Hurwitz numbers, and the expansion (2) can be identified as the partition function of a closed string theory with torus target space. The coupling and tension are given by $g_s = \frac{1}{N}$ and $T = \frac{1}{2\pi\alpha'} = \frac{\lambda}{A}$.

One of the most interesting properties of the QCD_2 string partition function is that the contributions (2) are quasi-modular forms on the elliptic curve whose Kähler class is dual to the modulus $\tau = -\frac{\lambda}{2\pi\,\mathrm{i}}$, i.e. they are polynomials in the basic holomorphic Eisenstein series $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$, with $F_h^+(\tau)$, $h \geq 2$ of weight $6h-6$ under the full modular group $PSL(2,\mathbb{Z})$ [8, 10]. This was first observed in [8] by direct inspection of the Feynman diagram expansion for the free energy within the conformal field theory approach to large N Yang-Mills theory proposed in [3]. A rigorous proof was subsequently presented in [10] directly from the equivalent free fermion representation of the partition function. The quasi-modular character of the $F_h^+(\tau)$ and their computability through the Feynman diagram expansion in a Kodaira-Spencer field theory confirm two general predictions of the mirror symmetry program in the special case of elliptic curves [11].

Explicit formulas for the free energy contributions up to genus $h = 8$ are given in [8] (see also [12]). In a suitable basis for the modular forms, they generally assume the forms

$$\begin{aligned} F_1^+(\lambda) &= -\epsilon_{\mathrm{F}} - \ln\eta(\tau)\ , \\ F_h^+(\lambda) &= \frac{\lambda^{2h-2}}{(2h-2)!\,\rho_h} \sum_{k=0}^{3h-3} \ \sum_{\substack{l,m\geq 0 \\ 2l+3m=3h-3-k}} s_{kl}\ E_2(\tau)^k\, E_2'(\tau)^l\, E_2''(\tau)^m \end{aligned} \tag{3}$$

with $h \geq 2$ and $\rho_h, s_{kl} \in \mathbb{N}$, where $\epsilon_{\mathrm{F}} = -\frac{\lambda}{12}\,N^2(N^2-1)$ is the Fermi energy and $\eta(\tau)$ is the Dedekind function. By using the modular transformation properties of the quasi-modular forms appearing in (3), it is possible to cast the string representation (2) in the weak-coupling regime $\lambda \to 0$ of the gauge theory and analyse the Douglas-Kazakov type singularity at zero area whereby a condensation of instantons in the vacuum occurs [13]. It was argued in [8] that the contributions are of the form

$$F_h^+(\lambda) = \lambda^{2h-2} \sum_{k=3h-3}^{4h-3} \frac{r_{k,h}\ \pi^{2(k-3h+3)}}{\lambda^k} + O\left(\mathrm{e}^{-1/\lambda}\right) \tag{4}$$

where $r_{k,h} \in \mathbb{Q}$ are related to the simple Hurwitz numbers. The structure of the small area singularities is very peculiar and it is not the most general one expected from the quasi-modular behaviour. In this paper we will elucidate the geometrical meaning of the rational numbers $r_{k,h}$. Explicit expressions up to genus $h = 6$, reflecting this structure, are found in [8].

3. Saddle-point solution. The proper setting for the study of weak-coupling expansions of the sort (4) is the instanton expansion, which for $U(N)$ Yang-Mills theory on $\mathbb{T}^2$

is given by [14]

$$
\begin{aligned}
Z(\lambda,N) &= (-1)^N \ \mathrm{e}^{\,-\epsilon_{\rm F}} \sum_{\boldsymbol{\nu}\in\mathbb{N}_0^N\,:\,\sum_k k\,\nu_k=N} \ \prod_{k=1}^N \frac{(-1)^{\nu_k}}{\nu_k!}\,\left(\frac{\pi^2N}{k^3\lambda}\right)^{\nu_k/2} \\
&\qquad \times \sum_{\boldsymbol{q}\in\mathbb{Z}^{|\boldsymbol{\nu}|}} (-1)^{(N-1)\sum_k q_k} \ \exp\left[-\frac{\pi^2N}{\lambda}\,\sum_{l=1}^N\frac1l \ \sum_{j=1+\nu_1+\dots+\nu_{l-1}}^{\nu_1+\dots+\nu_l} q_j^2\right] \ , \qquad (5)
\end{aligned}
$$

where we have defined $\nu_0:=0$ and the total number of partition components $|\boldsymbol{\nu}|:=\nu_1+\dots+\nu_N$. In (5) it is understood that if some $\nu_k=0$ then $q_{1+\nu_1+\dots+\nu_{k-1}}=\dots=q_{\nu_1+\dots+\nu_k}=0$. This result is derived by an explicit Poisson resummation of the full (non-chiral) Migdal expansion of the gauge theory. It represents the exact semi-classical expansion of the gauge theory path integral, written as a sum over unstable instantons.

By defining the sequence of functions

$$
\Xi_k(\lambda) := \sum_{q=-\infty}^\infty (-1)^{(N-1)q} \ \mathrm{e}^{\,-\frac{\pi^2N}{\lambda}\,\frac{q^2}{k}} \qquad (6)
$$

and resolving the constraint on the sum over partitions $\boldsymbol{\nu}$ in (5) explicitly by using a contour integral representation, we arrive at the formula

$$
Z(\lambda,N) = \mathrm{e}^{\,-\epsilon_{\rm F}} \oint\frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z^{N+1}} \ \exp\left[-\sqrt{\frac{\pi^2N}{\lambda}}\,\sum_{k=1}^\infty\Xi_k(\lambda)\,\frac{(-z)^k}{k^{3/2}}\right] \qquad (7)
$$

where the contour of integration encircles the origin $z=0$ of the complex z-plane and lies in the open unit disc $|z|<1$ in order to ensure convergence of the integrand. This representation of the weak-coupling partition function will be the key to extracting its form in the desired large N scaling limits. It can also be used to derive a concise resummation of the Migdal expansion. Applying the Poisson resummation formula to the sequence of functions (6), we may sum the series over k explicitly and bring (7) into the equivalent strong-coupling form

$$
Z(\lambda,N) = \mathrm{e}^{\,-\epsilon_{\rm F}} \oint\frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z^{N+1}} \ \prod_{n=-\infty}^\infty \left(1+z \ \mathrm{e}^{\,-\frac{\lambda}{N}\,(n-n_{\rm F})^2}\right) \ , \qquad (8)
$$

where $n_{\rm F}=\frac{N-1}{2}$ is the Fermi level. The contour integration now implements the constraint that the number of rows in the Young diagrams be bounded from above by the rank N of the gauge group. Eq. (8) agrees with the representation derived in [15] by direct group theory arguments and it is also similar to the contour integral formula obtained in [3] from the free fermion representation of the partition function.

We will now compute the partition function (7) in the large N limit with the 't Hooft coupling constant $\lambda=g^2A\,N/2$ held fixed, and compare it to the string representation of the previous section. In the weak coupling limit, the higher instanton contributions to the function (6) are of order $\mathrm{e}^{\,-N/\lambda}$ and hence can be neglected to a first-order approximation

at $N \to \infty$. We may thereby focus on the zero-instanton sector with vanishing magnetic charge $q = 0$, and write the vacuum amplitude as

$$\mathcal{Z}(\lambda, N) = \mathrm{e}^{-\epsilon_{\mathrm{F}}} \oint \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z} \exp\left[-N \ln z - \sqrt{\tfrac{\pi N}{\lambda}}\, \mathrm{Li}_{3/2}(-z)\right] , \tag{9}$$

where generally $\mathrm{Li}_\alpha(z) := \sum_{k\in\mathbb{N}} z^k/k^\alpha$ denotes the polylogarithm function of index α. We will compute the integral (9) in the large N limit by means of saddle-point techniques. The saddle-point equation is

$$\mathrm{Li}_{1/2}(-z) = -\sqrt{\tfrac{N\lambda}{\pi}} . \tag{10}$$

The function $\mathrm{Li}_{1/2}(-z)$ is a slowly decreasing negative function for $z \geq 1$, behaving as $\mathrm{Li}_{1/2}(-z) \simeq -2\sqrt{\ln(z)/\pi}$ for z real with $z \to \infty$. Thus a solution to (10) always exists and is located at large $z \in [-1, \infty)$.

The saddle-point equation (10) can be solved to any order in N by using the large x asymptotic expansion of the polylogarithm function [9]

$$\mathrm{Li}_\alpha\left(-\mathrm{e}^{x}\right) = 2\sum_{k=0}^{\infty} \frac{\left(1-2^{2k-1}\right) B_{2k}\, \pi^{2k}}{(2k)!\, \Gamma(\alpha+1-2k)}\, x^{\alpha-2k} , \tag{11}$$

where $B_{2k} \in \mathbb{Q}$ are the Bernoulli numbers. Corrections to this formula are of order e^{-x}. We seek a solution of the form $z = z_* = \mathrm{e}^{x_*}$ with x_* admitting an asymptotic $\frac{1}{N}$ expansion

$$x_* = x_{-1}\, N + \sum_{k=0}^{\infty} \frac{x_k}{N^k} . \tag{12}$$

One can now proceed to obtain the coefficients x_k of the saddle-point solution recursively in powers of $\frac{1}{N}$ by substituting (12) and (11) into (10), and the result of a straightforward iterative evaluation up to order $1/N^{11}$ reads

$$x_* = \tfrac{N\lambda}{4} + \tfrac{\pi^2}{3N\lambda} + \tfrac{16\pi^4}{9(N\lambda)^3} + \tfrac{448\pi^6}{9(N\lambda)^5} + \tfrac{1254656\pi^8}{405(N\lambda)^7} + \tfrac{406598656\pi^{10}}{1215(N\lambda)^9} + \tfrac{67556569088\pi^{12}}{1215(N\lambda)^{11}} + O\left(\tfrac{1}{N^{13}}\right) . \tag{13}$$

The systematic vanishing of the x_k for even powers of $\frac{1}{N}$ is exactly what is expected from the form of the Gross-Taylor expansion.

Armed with the solution (13) of the saddle-point equation, we can now proceed to compute the free energy $\mathcal{F}(\lambda, N) = \ln \mathcal{Z}(\lambda, N)$ by parametrizing the integration variable in (9) as $z = \mathrm{e}^{x_*+x}$, where the variable x contains contributions from fluctuations about the saddle-point value which can be integrated over all $x \in \mathbb{R}$ since the deviations from the results computed with the correct integration domain appropriate to (9) will be exponentially suppressed in the small area limit. By using (11), a numerical evaluation up

to order $1/N^{10}$ using *Mathematica* yields

$$\begin{aligned}\mathcal{F}(\lambda,N) &= \tfrac{\ln(-2\pi\,\lambda)}{2}+\tfrac{\pi^2}{6\lambda}+\left(\tfrac{2}{3\lambda}-\tfrac{2\pi^2}{3\lambda^2}+\tfrac{8\pi^4}{45\lambda^3}\right)\tfrac{1}{N^2}+\left(-\tfrac{8}{\lambda^2}+\tfrac{16\pi^2}{\lambda^3}-\tfrac{100\pi^4}{9\lambda^4}+\tfrac{224\pi^6}{81\lambda^5}\right)\tfrac{1}{N^4}\\ &+\left(\tfrac{2272}{9\lambda^3}-\tfrac{2272\pi^2}{3\lambda^4}+\tfrac{8096\pi^4}{9\lambda^5}-\tfrac{41504\pi^6}{81\lambda^6}+\tfrac{48256\pi^8}{405\lambda^7}\right)\tfrac{1}{N^6}\\ &+\left(-\tfrac{13504}{\lambda^4}+\tfrac{54016\pi^2}{\lambda^5}-\tfrac{834304\pi^4}{9\lambda^6}+\tfrac{7010816\pi^6}{81\lambda^7}-\tfrac{17887904\pi^8}{405\lambda^8}+\tfrac{11958784\pi^{10}}{1215\lambda^9}\right)\tfrac{1}{N^8}\\ &+\left(\tfrac{15465472}{15\lambda^5}-\tfrac{15465472\pi^2}{3\lambda^6}+\tfrac{105156608\pi^4}{9\lambda^7}-\tfrac{418657280\pi^6}{27\lambda^8}\right.\\ &\left.+\tfrac{572409344\pi^8}{45\lambda^9}-\tfrac{2467804672\pi^{10}}{405\lambda^{10}}+\tfrac{33778284544\pi^{12}}{25515\lambda^{11}}\right)\tfrac{1}{N^{10}}+O\left(\tfrac{1}{N^{12}}\right)\,. \end{aligned}\qquad (14)$$

The expression (14) matches *precisely* eqs. (3.43)–(3.47) in [8] which were obtained using the conformal field theory representation of large N two-dimensional Yang-Mills theory. The saddle-point evaluation of the zero-instanton sector thereby reproduces all non-exponentially suppressed terms in the weak-coupling expansion of the chiral $U(N)$ free energy, giving the correct rational numbers $r_{k,h}$ appearing in (4). An important ingredient in this reproduction is the cancellation of the ground state energy $\epsilon_{\rm F}$, since this term would otherwise dominate the series in the large N limit. In particular, the expansion starts at order N^0 and there is no spherical contribution, consistent with the fact that there are no unfolded coverings of a torus by a sphere. The expansion (14) also contains the correct leading modular dependence of the Dedekind function coming from the genus 1 free energy. Let us also point out three particularly noteworthy features of our derivation of the formula (14). First of all, it is a highly accurate check of the expansion obtained in [8] by a completely independent method. Secondly, it does not rely on any group theory or conformal field theory techniques and is obtained directly in the instanton representation. As far as we are aware, this is the first time that stringy quantities are computed in two dimensional Yang-Mills theory directly from the weak-coupling expansion, which is the most natural one from a field theoretic point of view. This is possible due to the absence of a large N phase transition at finite area on the torus, which would otherwise prohibit the recovery of stringy features from weak-coupling data [13]. Finally, we note that the dynamics of the zero-instanton sector is surprisingly rich, encoding properly the anticipated string characteristics. This is related to the underlying structure of the topological string theory governing the weak-coupling limit [7, 11].

However, despite this remarkable agreement, the computation above reproduces only the *chiral* part of the full $U(N)$ gauge theory. The saddle-point technique has not picked up the coupling to the anti-chiral contributions, representing the anti-holomorphic sector of the closed string theory. Part of the problem can be traced back to the fact that the argument of the exponential integrand in (9) does not admit a nice large N scaling. Moreover, we eventually have to face up to the problem of evaluating the non-zero higher instanton contributions. It is very likely that, in spite of their exponential suppression order by order in $\frac{1}{N}$, their collective behaviour will be crucial to recover the complete string expansion. The problem of how much information should be carried by the higher instantons in order to reproduce the full string partition function will be addressed elsewhere [9].

4. Double scaling limit. Motivated by the analysis of the previous section, we will now analyse the $U(N)$ gauge theory in a large N limit wherein the saddle-point technique

can capture the entire relevant story. We will take the limit $N \to \infty$ while keeping fixed the new scaled coupling constant $\mu := \frac{1}{\pi} N \lambda = \frac{1}{2\pi} N^2 g^2 A$. The partition function then assumes the form

$$\hat{\mathcal{Z}}(\mu, N) = \mathrm{e}^{\frac{\pi N \mu}{12}} \oint \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}\ \exp\left[-N\left(\ln z + \tfrac{1}{\sqrt{\mu}}\ \mathrm{Li}_{3/2}(-z)\right)\right] := \oint \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z}\ \mathrm{e}^{\,N\hat{F}(z,\mu)} \quad (15)$$

and hence has a nice large N limit. Corrections to this expression from higher instanton configurations are of order $\mathrm{e}^{-N^2/\mu}$ and could be completely suppressed in the $\frac{1}{N}$ expansion. In fact, at $N = \infty$ the vacuum amplitude is given by the leading planar term $\hat{\mathcal{Z}}(\mu, N) = \mathrm{e}^{\,N\hat{F}(z_*,\mu)}$ in the $\frac{1}{N}$ expansion of the integral (15), which can be rigorously computed in the saddle-point approximation. We refer to this new limit of the gauge theory as the "double scaling limit". It is very different from the conventional planar large N limit used to derive the Gross-Taylor expansion.

Starting from (15), we derive the saddle-point equation

$$\mathrm{Li}_{1/2}(-z) = -\sqrt{\mu}\ , \quad (16)$$

and we will show that it can be solved exactly. In contrast to the conventional 't Hooft limit, this equation does not depend on N and therefore its solution $z = z_*(\mu)$ does not rely on any approximation in general. It can be simply written as $z_*(\mu) = -\mathrm{Li}_{1/2}^{-1}(-\sqrt{\mu}\,)$, the inverse function being uniquely defined in the region of interest thanks to the monotonic behaviour of the polylogarithm function $\mathrm{Li}_{1/2}(-z)$. The double scaling free energy can then be evaluated as $\hat{\mathcal{F}}(\mu, N) = N\,\hat{F}(z_*(\mu), \mu)$. We can write down an *exact* relation that gives it directly as a primitive of the position of the saddle point. By parametrizing the saddle point solution as before in the form $z_*(\mu) = \mathrm{e}^{\,x_*(\mu)}$ and defining $y := \sqrt{\mu}$, one can easily derive the equation

$$\hat{F}\big(\mathrm{e}^{\,x_*(\mu)}\,,\,\mu\big) = \frac{1}{\sqrt{\mu}}\int_{\sqrt{\mu}}^{\infty}\mathrm{d}y\ \left[x_*\left(y^2\right) - \tfrac{\pi}{4}\,y^2\right]\ . \quad (17)$$

This explicit representation is useful because it exhibits the structure of the free energy straightforwardly in terms of the properties of the saddle point solution. It is a smooth function of $\mu > 0$, with a logarithmic singularity in the weak-coupling limit $\mu \to 0$ where the Douglas-Kazakov type phase transition takes place.

The physical meaning of the double scaling limit can be understood by expanding the free energy for large μ. It is in this regime that we expect to see a relation with the string picture derived previously in the 't Hooft scaling limit. The strong-coupling saddle point solution is given by the expansion (13) which is naturally written as a series in the double scaling parameter given by

$$x_*(\mu) = \pi\sum_{k=0}^{\infty}\frac{\xi_{2k-1}}{\mu^{2k-1}}\ . \quad (18)$$

The double scaling free energy may then be computed directly from (17) to get

$$\hat{\mathcal{F}}(\mu, N) = \pi\,N\left[\sum_{k=1}^{\infty}\frac{\xi_{2k-1}}{4k-3}\ \frac{1}{\mu^{2k-1}} + O\left(\mathrm{e}^{-\mu}\right)\right]\ , \quad (19)$$

where we note the cancellation of the vacuum energy contribution. From the explicit expression in (13) the first few terms are found to be given by

$$\hat{\mathcal{F}}(\mu,N)=2\pi\,N\,\left[\tfrac{1}{6\mu}+\tfrac{8}{45\mu^3}+\tfrac{224}{81\mu^5}+\tfrac{48256}{405\mu^7}+\tfrac{11958784}{1215\mu^9}+\tfrac{33778284544}{25515\mu^{11}}+O\left(\tfrac{1}{\mu^{13}}\right)+O\left(\mathrm{e}^{-\mu}\right)\right]\,. \tag{20}$$

Comparing with (14), we see that at strong-coupling the double scaling limit has extracted the most singular terms, as $\lambda\to 0$, at each order of the original $\frac{1}{N}$ expansion. In other words, the double scaled gauge theory at strong-coupling presents a resummation of the most singular terms in the weak-coupling limit of the chiral Gross-Taylor string expansion. The leading contribution to (4) in the double scaling limit at $N=\infty$ is given at $k=4h-3$, and thus the general form of this expansion can be written as in (19) with the rational numbers $r_{4h-3,h}$ completely determined by the strong-coupling solution (18) of the saddle point equation as $r_{4h-3,h}=\xi_{2h-1}/(4h-3)$.

Let us now elucidate the geometrical meaning of the rational numbers $r_{4h-3,h}$. The $\lambda\to 0$ behaviour of the series (2) is controlled by the large n asymptotics of the simple Hurwitz numbers ω_h^n. At fixed genus h, the singularities of the free energy (2) as $\lambda\to 0$ are related to a power-like growth $\omega_h^n\simeq\beta_h\,n^{\alpha_h}$ of the number of holomorphic branched covering maps of $\mathbb{T}^2$ with large winding number n, where $\alpha_h>0$ and

$$\beta_h=(\alpha_h+1)\,\lim_{N\to\infty}\,\frac{1}{N^{\alpha_h+1}}\,\sum_{n=1}^N\omega_h^n\ . \tag{21}$$

The leading singularity of the series (2) as $\lambda\to 0$ is extracted by substituting these asymptotics to get

$$\lim_{\lambda\to 0}\,F^+(\lambda,N)=\sum_{h=1}^\infty\left(\frac{\lambda}{N}\right)^{2h-2}\,\beta_h\,\mathrm{Li}_{-\alpha_h}\big(\mathrm{e}^{-\lambda}\big)\ . \tag{22}$$

In the limit $\lambda\to 0$, we can substitute the leading singular behaviour of the polylogarithm function $\mathrm{Li}_{-\alpha_h}(z)\simeq\Gamma(\alpha_h+1)\,(-\ln z)^{-\alpha_h-1}$ for $z\to 1^-$ [16]. Matching this to the expansion (19) of the free energy with $\lambda=\pi\,\mu/N$, we find the power of the growth as the natural number $\alpha_h=4h-4$, while the positive numbers β_h are given by $\beta_h=\xi_{2h-1}\,\pi^{2h}/(4h-3)!$. That the explicit knowledge of the small area behaviour of the gauge theory allows one to reconstruct the asymptotic forms of the simple Hurwitz numbers is our first main result.

Proposition 1 *The asymptotic expansion as $\mu\to\infty$ for the free energy (19) of $U(N)$ gauge theory on $\mathbb{T}^2$ in the double-scaling limit is the generating function for the asymptotic Hurwitz numbers with*

$$\lim_{n\to\infty}\,\omega_h^n=\frac{\pi^{2h}}{(4h-3)!}\,\xi_{2h-1}\,n^{4h-4}\ .$$

Thus the saddle-point equation for the zero-instanton double scaling free energy solves the combinatorial problem of determining the asymptotics of Hurwitz numbers. We will now compute the explicit forms of the coefficients of the saddle-point expansion (18,19) as polynomials in Bernoulli numbers, and thereby write down the *exact* solution of the

double scaling gauge theory in the strong-coupling limit. For this, we set $x_* = 1/w^2$ and use (11) to write the saddle-point equation (16) as

$$w = -\frac{2}{\sqrt{\mu}} \sum_{k=0}^{\infty} \frac{\left(1-2^{2k-1}\right) B_{2k}\ \pi^{2k}}{(2k)!\ \Gamma\left(\frac{3}{2}-2k\right)}\ w^{4k} := \frac{1}{\sqrt{\mu}}\ L(w)\ . \tag{23}$$

The solution $w(\mu)$ of (23) can be found by means of the Lagrange inversion formula. For our purposes it will be more useful to employ the Burmann generalization, which may be represented succinctly as the contour integral

$$G\big(w(\mu)\big) = \oint \frac{\mathrm{d}z}{2\pi\,\mathrm{i}}\ G(z)\ \frac{1-\frac{1}{\sqrt{\mu}}\,L'(z)}{z-\frac{1}{\sqrt{\mu}}\,L(z)} \tag{24}$$

where $G(z)$ is any analytic function. When $G(z) = z$, the formal Taylor series expansion of the integrand of (24) in powers of $\frac{1}{\sqrt{\mu}}$ reproduces the usual Lagrange inversion formula. In our case, we should take $G(z) = 1/z^2$, but this cannot be directly inserted into the formula (24) as it would introduce a spurious contribution from the double pole at $z = 0$. The simplest way to deal with this problem is to subtract the undesired contribution by hand, and thereby write the solution $x_*(\mu)$ of the saddle-point equation as

$$x_*(\mu) = \oint \frac{\mathrm{d}z}{2\pi\,\mathrm{i}\,z^2}\ \frac{1-\frac{1}{\sqrt{\mu}}\,L'(z)}{z-\frac{1}{\sqrt{\mu}}\,L(z)} - \frac{L(0)\,L''(0)-\big(\sqrt{\mu}-L'(0)\big)^2}{L(0)^2}\ . \tag{25}$$

By formally expanding (25) in powers of $\frac{1}{\sqrt{\mu}}$, we arrive at the strong-coupling solution

$$x_*(\mu) = -\frac{L(0)\,L''(0)-\big(\sqrt{\mu}-L'(0)\big)^2}{L(0)^2} - 2\sum_{k=1}^{\infty} \frac{1}{(k+2)!\,k}\ \frac{\mathrm{d}^{k+2}}{\mathrm{d}z^{k+2}} L(z)^k\bigg|_{z=0}\ \frac{1}{\mu^{k/2}}\ . \tag{26}$$

From the definition of the function $L(z)$ in (23), one finds that the coefficients of the series in (26) are non-zero only when $k = 4h-2$ for some $h \in \mathbb{N}$. Written in the form (18), one computes $\xi_{-1} = \frac{1}{4}$ and

$$\xi_{2h-1} = -\frac{\pi^{2h-1}}{2h-1} \sum_{\boldsymbol{h}\in\mathbb{N}_0^{4h-2}\,:\,\sum_k h_k=h}\ \prod_{k=1}^{4h-2} \frac{\left(2-2^{2h_k}\right)\ B_{2h_k}}{(2h_k)!\ \Gamma\left(\frac{3}{2}-2h_k\right)} \tag{27}$$

for $h \geq 1$, and we have thereby found the complete strong-coupling expansion of the double scaled gauge theory. We can simplify the sum over ordered partitions $\boldsymbol{h} \in \mathbb{N}_0^{4h-2}$ of the integer h by reducing it to a sum over partitions of h into m positive integers. By inserting 0 into all possible positions we obtain $\binom{4h-2}{m}$ partitions of the original type in (27), and we find

$$\begin{aligned}
\xi_{2h-1} &= -\frac{\pi^{2h-1}}{2h-1} \sum_{m=1}^{h} \binom{4h-2}{m} \left(\frac{B_0}{\Gamma\left(\frac{3}{2}\right)}\right)^{4h-2-m} \sum_{\substack{\boldsymbol{h}\in\mathbb{N}^m \\ \sum_k h_k=h}}\ \prod_{k=1}^{m} \frac{\left(2-2^{2h_k}\right)\ B_{2h_k}}{(2h_k)!\ \Gamma\left(\frac{3}{2}-2h_k\right)} \\
&= -\sum_{m=1}^{h} \frac{(-1)^m\,2^{2h+m-1}\,(4h-3)!}{(4h-2-m)!\,m!} \sum_{\substack{\boldsymbol{h}\in\mathbb{N}^m \\ \sum_k h_k=h}}\ \prod_{k=1}^{m} \frac{\left(2^{2h_k-1}-1\right)\ (4h_k-3)!!\ B_{2h_k}}{(2h_k)!}\ .
\end{aligned} \tag{28}$$

Finally, by exploiting the symmetry of the second summand in (28) we can reduce the sum over ordered partitions of h with m components to a sum over conjugacy classes and cycles of the symmetric group S_h, i.e. over unordered partitions of h. An unordered partition of h is specified by h non-negative integers ν_k with $\sum_k k\,\nu_k = h$, while the condition that the partition contain only m parts is implemented by requiring that $\sum_k \nu_k = m$. By inserting the combinatorial factor $\frac{m!}{\nu_1!\cdots\nu_h!}$ which counts the number of different ordered partitions that originate from the same unordered partition, we may bring (28) into our final equivalent form.

Theorem 1 *The coefficients of the strong-coupling saddle-point expansion for the free energy (19) of $U(N)$ gauge theory on $\mathbb{T}^2$ in the double-scaling limit are given by*

$$\begin{aligned}\xi_{2h-1} &= (4h-3)!\,\sum_{m=1}^{h}\frac{(-1)^{m-1}\,2^{m+2h-1}}{(4h-2-m)!} \\ &\quad\times \sum_{\substack{\boldsymbol{\nu}\in\mathbb{N}_0^h \\ \sum_k k\,\nu_k=h\,,\,\sum_k\nu_k=m}}\ \prod_{k=1}^{h}\frac{1}{\nu_k!}\left(\frac{\left(2^{2k-1}-1\right)\,(4k-3)!!\;B_{2k}}{(2k)!}\right)^{\nu_k}\ .\end{aligned}$$

This formula coincides *precisely* with Theorem 7.1 of [17], whereby the asymptotics of simple Hurwitz numbers are evaluated directly by involved combinatorial techniques. Here we have shown that the saddle-point equation provides a very efficient and much simpler method for extracting these numbers.

5. Principal moduli spaces of holomorphic differentials. As we will now demonstrate, Proposition 1 implies that the double scaling limit of the $U(N)$ gauge theory on $\mathbb{T}^2$ is intimately related to the geometry of some very special moduli spaces [17, 18], well-known in ergodic theory, whose points are "counted" by the strong coupling expansion coefficients of the previous section. Let $\mathcal{M}_h$ be the moduli space of (topological classes of) pairs $(\Sigma,\mathrm{d}\boldsymbol{u})$, where Σ is a compact Riemann surface of genus h and $\mathrm{d}\boldsymbol{u}$ is a holomorphic one-form on Σ with exactly $2h-2$ simple zeroes $\{u_i\}_{i=1}^{2h-2}\subset\Sigma$. We call $\mathcal{M}_h$ a principal moduli space of holomorphic differentials. It can be coordinatized as follows. Consider the relative homology group $H_1(\Sigma,\{u_i\};\mathbb{Z})\cong\mathbb{Z}^{4h-3}$, and choose a basis of relative one-cycles $\{\gamma_i\}_{i=1}^{4h-3}$ such that γ_i for $i=1,\dots,2h$ form a canonical symplectic basis of one-cycles for the ordinary homology group $H_1(\Sigma;\mathbb{Z})$, while the open contours γ_{2h+i} for $i=1,\dots,2h-3$ connect the zeroes u_{i+1} to u_1 on Σ. We define the corresponding period map $\phi:\mathcal{M}_h\to\mathbb{C}^{4h-3}$ by $\phi(\Sigma,\mathrm{d}\boldsymbol{u}) := (\int_{\gamma_1}\mathrm{d}\boldsymbol{u}\,,\ \dots\ ,\int_{\gamma_{4h-3}}\mathrm{d}\boldsymbol{u})$. This map defines a local system of complex coordinates on $\mathcal{M}_h$ which makes it a complex orbifold of dimension $\dim\mathcal{M}_h = 4h-3$. By using the period map we can also define a smooth measure on the moduli space $\mathcal{M}_h$ using the pull-back of the Lebesgue measure $\mathrm{d}\nu$ from $\mathbb{C}^{4h-3}$ to $\mathcal{M}_h$ under ϕ. However, the total volume of $\mathcal{M}_h$ with respect to this measure is infinite. To cure this, we restrict to the subspace $\mathcal{M}_h'\subset\mathcal{M}_h$ consisting of pairs $(\Sigma,\mathrm{d}\boldsymbol{u})$ such that the area of the surface Σ is 1 with respect to the metric defined by the holomorphic one-form $\mathrm{d}\boldsymbol{u}$. Let $\mathrm{C}_N\phi\,(\mathcal{M}_h') := \{t\,\phi(\mathcal{M}_h')\mid 0\le t\le\sqrt{N}\,\}$ be a sequence of cones over $\phi(\mathcal{M}_h')$ with

vertex at the origin of $\mathbb{C}^{4h-3}$. We may then define the finite volume of the moduli space $\mathcal{M}'_h$ by the formula

$$\mathrm{vol}\,(\mathcal{M}'_h) := \int_{\mathrm{C}_1\phi(\mathcal{M}'_h)} \mathrm{d}\nu \ 1 \ . \tag{29}$$

Let us describe the relationship between these moduli spaces and the double scaling limit of two-dimensional Yang-Mills theory on the torus $\mathbb{T}^2$. Consider a branched covering $\varpi : \Sigma \to \mathbb{T}^2$ of the torus by a Riemann surface of genus h, with simple ramification over distinct points $\{z_i\}_{i=1}^{2h-2} \subset \mathbb{T}^2$. With $\mathrm{d}z$ denoting the canonical holomorphic differential on $\mathbb{T}^2$, one can use the pull-back under the covering map to associate the point $(\Sigma\,,\,\varpi^*(\mathrm{d}z)) \in \mathcal{M}_h$ with simple zeroes $u_i = \varpi^{-1}(z_i)$ corresponding to the ramification points of the cover. Conversely, given $(\Sigma, \mathrm{d}\boldsymbol{u}) \in \mathcal{M}_h$ we can define a covering map $\varpi : \Sigma \to \mathbb{T}^2$ by $z = \varpi(u) := \int^u \mathrm{d}\boldsymbol{u} \mod \mathbb{Z}^2$. The critical points of ϖ are precisely the simple zeroes u_i of the holomorphic differential $\mathrm{d}\boldsymbol{u} = \varpi^*(\mathrm{d}z)$, and ϖ thereby has simple ramification at $u_i \in \Sigma$. The degree of this covering map is the area of Σ with respect to the metric defined by $\mathrm{d}\boldsymbol{u}$. We have thereby arrived at a one-to-one correspondence between simple branched covers of the torus, and hence terms in the chiral Gross-Taylor string expansion, and points in the principal moduli spaces of holomorphic differentials.

We will now show that the numbers (29) are computed by the strong-coupling saddle-point expansion of the gauge theory that we obtained in the previous section. The basic idea is that the counting of points $(\Sigma\,,\,\varpi^*(\mathrm{d}z)) \in \mathcal{M}_h$ is like counting lattice points $\mathbb{Z}^{2(4h-3)}$ inside subsets of $\mathbb{R}^{2(4h-3)} \cong \mathbb{C}^{4h-3}$. Using the definition (29) we may compute the volumes of the principal moduli spaces as

$$\mathrm{vol}\,(\mathcal{M}'_h) = \lim_{N\to\infty} \tfrac{1}{N^{4h-3}} \left| \mathrm{C}_N\phi\,(\mathcal{M}'_h) \ \cap \ \big(\mathbb{Z}^{2(4h-3)} + \boldsymbol{b}\big) \right| \ , \tag{30}$$

where the vector $\boldsymbol{b} = (b_i) \in \mathbb{C}^{4h-3}$ has components $b_i \in \mathbb{Z}^2$ for $i = 1,\dots,2h$ while $b_i \neq b_j \mod \mathbb{Z}^2$ for $i,j > 2h$ with $i \neq j$. On the other hand, from the above correspondence it follows that each point of the intersection $\mathrm{C}_N\phi(\mathcal{M}'_h) \cap (\mathbb{Z}^{2(4h-3)} + \boldsymbol{b})$ corresponds to a simple branched cover ϖ of $\mathbb{T}^2$ with winding number $\leq N$, and thus the volume (30) may be computed via the asymptotics of simple Hurwitz numbers as

$$\mathrm{vol}\,(\mathcal{M}'_h) = \lim_{N\to\infty} \frac{1}{N^{4h-3}} \sum_{n=1}^{N} \omega_h^n \ . \tag{31}$$

Comparing with (21) and Proposition 1 we thereby find that the volumes are completely determined in terms of the coefficients ξ_{2h-1} of the saddle-point solution to be

$$\mathrm{vol}\,(\mathcal{M}'_h) = \frac{\xi_{2h-1}}{(4h-3)!\ (4h-3)} \ \pi^{2h} \ . \tag{32}$$

Using the formula (32) and the explicit expansion (20), the first few volumes can be readily computed and are summarized in Table 1. The general combinatorial solution is provided by Theorem 1. In particular, the saddle-point computation explicitly demonstrates the rationality property $\pi^{-2h}\ \mathrm{vol}(\mathcal{M}'_h) \in \mathbb{Q}$ [17, 18] and gives a precise geometrical meaning to the rational numbers that we first encountered in the weak-coupling expansions of Section 2.

h	$\mathrm{vol}(\mathcal{M}'_h)/\pi^{2h}$
1	$\frac{1}{3}$
2	$\frac{2}{675}$
3	$\frac{1}{65610}$
4	$\frac{29}{757795500}$
5	$\frac{23357}{422031469860000}$
6	$\frac{16493303}{318258151736124600000}$

Table 1: *The normalized volumes of the principal moduli spaces of holomorphic differentials up to genus $h = 6$.*

Proposition 2 *The asymptotic expansion as $\mu \to \infty$ for the free energy of $U(N)$ gauge theory on $\mathbb{T}^2$ in the double-scaling limit is the generating function for the volumes of the principal moduli spaces of holomorphic differentials given by*

$$\hat{\mathcal{F}}(\mu, N) = N \sum_{h=1}^{\infty} \frac{(4h-3)!}{(\pi\,\mu)^{2h-1}} \,\mathrm{vol}\left(\mathcal{M}'_h\right) + O\left(\mathrm{e}^{-\mu}\right) \ .$$

6. String theory interpretation. Let us conclude by briefly discussing some of the implications that our results have on the string expansion of two-dimensional Yang-Mills theory. As we have seen, only the chiral part of the Gross-Taylor series contributes in the double scaling limit of $U(N)$ gauge theory on $\mathbb{T}^2$, and so the effective string theory is necessarily chiral. Alternatively, we may regard this fact as being the statement that the holomorphic and anti-holomorphic sectors are identified with one another, so that the expansion is in terms of *open* strings. Being a chiral series, the original strong-coupling expansion is given as a sum over *all* Young diagrams, labelling representations of the infinite unitary group $U(\infty)$, which is the gauge group of a *noncommutative* gauge theory. In the strong coupling regime, the double scaling limit produces the asymptotic Hurwitz numbers, of branched covering maps to $\mathbb{T}^2$ in the limit of infinite winding number, again reflecting the open nature of the string degrees of freedom. This latter feature also implies that the toroidal spacetime is effectively decompactified onto the plane in the double scaling limit, corresponding to the equi-anharmonic limit $\tau \to \mathrm{i}\,\infty$ of the underlying elliptic curve. All of these features suggest that the proper setting for understanding the physics of the double scaling limit is through the fluxon expansion of gauge theory on the noncommutative plane. This is indeed the case and will be analysed in more detail in [9].

Let us now note that the Gross-Taylor series is an expansion in $\frac{1}{N}$, i.e. it is *perturbative* in the string coupling g_s, and as such it ignores non-perturbative corrections of the form $\mathrm{e}^{-N\,\lambda}$. These contributions have a natural interpretation [19] as coming from (Euclidean) D1-branes of tension $T_1 = \frac{1}{\pi\,\alpha' g_s}$ which wrap around the target space torus $\mathbb{T}^2$ without foldings. The new double scaling limit of this paper captures the D-string contributions which are of order $\mathrm{e}^{-\mu}$, with the double scaled coupling constant related to the brane tension. These contributions play an important role in ensuring regularity of the complete free energy given by (17), as their collective behaviour must eliminate the divergences

which are just artifacts of the strong-coupling approximation. We can now rewrite the strong-coupling limit of the free energy (19) in terms of open string parameters as

$$\mathcal{Z}_{\rm str}(g_s, T_1 A) := \lim_{\mu\to\infty} \hat{\mathcal{F}}(\mu, N) = \frac{1}{g_s} \sum_{k=1}^{\infty} 2^{2k-1}\,(4k-3)!\,\frac{\mathrm{vol}\,(\mathcal{M}'_k)}{(T_1 A)^{2k-1}}\ . \tag{33}$$

We interpret this expansion as a remnant of the resummation of the D1-branes provided by (17). The effective action (33) is of order $1/g_s$, thereby representing an open string *disk* amplitude, and so the double scaling limit can be interpreted as a theory of D1-branes at tree-level in open string perturbation theory. The truncation of the dynamics to tree level is presumably related to the fact that the strong coupling regime of the double scaling gauge theory is described by some sort of topological open string theory. The natural appearence of moduli space volumes strongly suggests that an explicit realization of the string partition function generically as an Euler sigma-model should be possible, so that the string path integral localizes in the usual way onto finite dimensional moduli spaces [7]. The double scaling limit replaces the counting of holomorphic maps $\Sigma \to \mathbb{T}^2$, arising in the 't Hooft limit of QCD_2 on the torus, by the counting of holomorphic differentials on complex curves Σ. The action principle for this open string theory is given by the topologically twisted $\mathcal{N}=2$ superconformal field theory coupled to gravity that describes the closed string expansion of two-dimensional Yang-Mills theory [11], by taking the torus target space to be equi-anharmonic after a modular transformation. More details will be given in [9].

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